Gravitational Stabilization of a Satellite in a Fixed Inertial Orientation

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For satellites with long operational lifetimes it is often necessary to avoid mass expulsion techniques for attitude control. One alternative, which has been used successfully for Earth-pointing satellites, is the exploitation of gravitational forces exerted on the satellite by the central body. This paper deals with the possibility of using such forces to stabilize a satellite in an *inertially fixed* orientation. In the proposed attitude control scheme, the inertia properties of the satellite are selected in such a way that the gravitational forces have a natural stabilizing effect on motions about axes which lie in the orbit plane. To stabilize motions about an axis normal to the orbit plane (the pitch axis), an active control device is included. This device varies the inertia properties of the system by moving masses within the satellite so as to produce a controlled gravitational torque about the pitch axis when there is a pitch attitude error. The theoretical feasibility of the proposed control scheme is demonstrated by showing that there are regions in the system parameter space for which the desired attitude motion is asymptotically stable.

Introduction

ARTIFICIAL satellites used for such purposes as communications and navigation can have operational lifetimes measured in years. In selecting attitude control systems for such satellites, it is desirable, or even essential, to avoid the use of mass-expulsion techniques because the weight of the required fuel supply and associated hardware can be excessive. One attractive alternative is to utilize environmental forces such as gravitational or electromagnetic forces which has been done successfully in controlling the attitude of several Earth pointing satellites.[1] This paper demonstrates the theoretical feasibility of using gravitational forces to stabilize a satellite in an *inertially fixed* orientation. A need for such satellites arises, for example, in connection with astronomical observations.

To demonstrate this idea, a particular satellite is analyzed. The satellite has an axisymmetric inertia ellipsoid for its mass center when the satellite has the desired inertially fixed orientation; and the axis of symmetry, or pitch axis, is then normal to the orbit plane. For this configuration, the gravity torque has a natural stabilizing effect on motions about the transverse or roll-yaw axes, and viscous dampers are used to dissipate the energy of undesired oscillations. The satellite includes an active control device which varies the inertia properties of the system by moving masses within the satellite so as to produce a controlled gravity torque about the pitch axis when there is a pitch attitude error.

System Selection

For a satellite which is to maintain a fixed inertial orientation, it is important that the design be such that there are no disturbance torques due to gravity when the satellite has the desired orientation. In order to determine the restrictions placed on the system by this requirement, consider a satellite S whose mass center S^* moves in a circular orbit around a central body E as pictured in Fig. 1. The gravity torque, that is, the sum of the moments about S^* of all gravitational forces exerted on S by E, depends on the mass distribution of E, which is assumed to be spherically symmetric.

If S has principal moments of inertia I_1, I_2, and I_3 for S^*, and if $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ are unit vectors parallel to the corresponding principal axes, then the gravity torque $\mathbf{T}$ can be expressed as[2]

$$\mathbf{T} = 3n^2[(I_3 - I_2)c_3c_2\mathbf{b}_1 + (I_1 - I_3)c_1c_3\mathbf{b}_2 + (I_2 - I_1)c_2c_1\mathbf{b}_3] \tag{1}$$

where n, the mean motion is defined in terms of the orbital period T as

$$n = 2\pi/T \tag{2}$$

and c_1, c_2, and c_3 are the cosines of the angles between the line ES^* and $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$, respectively.

Inspection of Eq. (1) shows that $\mathbf{T}$ will vanish if any two direction cosines are zero, if all three moments of inertia are equal, or if two moments of inertia are equal and one direction cosine is equal to zero (e.g., $I_1 = I_2$, $c_3 = 0$). The first case corresponds to an Earth-pointing satellite (the traditional application of gravity stabilization) and does not apply to the problem at hand. In the second case, the inertia ellipsoid of the satellite is spherical and the gravity torque vanishes regardless of the satellite's orientation. The last set of conditions corresponds to an axisymmetric satellite which is oriented so that the axis of symmetry is normal to the orbit plane. Because the symmetric satellite allows for passive gravity control about two axes (as will be discussed), this configuration has been selected.

The next question to be considered is that of the attitude stability of the satellite when it is in the desired orientation. The question of the effect of the gravity torque on motions

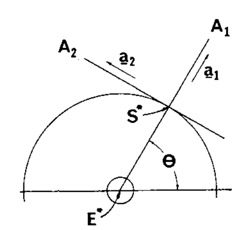

Fig. 1 Satellite orbit.

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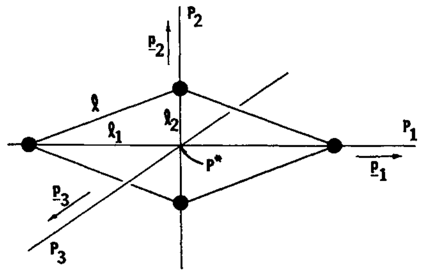

Fig. 2 Pitch controller.

about the transverse or roll-yaw axes of a symmetrical satellite has been partially answered by the analyses of Thomson[3] and Kane,[4] who studied the effect of gravity torque on the roll-yaw stability of a symmetrical rigid body in a circular orbit when the body is spinning about its symmetry axis which is normal to the orbit plane. Their results show that for the special case corresponding to a fixed inertial orientation (zero spin rate) there is a range of values of the ratio of axial to transverse moment of inertia for which the roll-yaw attitude is infinitesimally stable. That is, the solution of the linearized system equations is marginally stable. This suggests that, with the addition of some sort of passive energy-dissipating device, or damper, which does not destroy the symmetry of the system, the gravity torque might provide asymptotic roll-yaw stability in a completely passive manner.

Examination of Eq. (1) makes it clear that for a symmetric satellite there is no component of the gravity torque along the axis of symmetry, or pitch axis, regardless of the orientation of the satellite. As a consequence, undesirable motion about this axis will be unaffected by the gravity torque unless a device is added which can vary the inertia properties so as to destroy the symmetry when there is a pitch attitude error. Such a control device will necessarily be active in that it will require both sensors to detect attitude errors and power input to alter the inertia properties.

On the basis of these observations, the satellite system which has been selected for analysis incorporates an active pitch controller and two passive roll-yaw dampers. The dampers consist of symmetric rotors the axes of which are fixed in the main body and whose motions relative to the main body are resisted by viscous torques. Pitch damping is obtained by utilizing rate feedback.

The pitch controller, designated P, is pictured in Fig. 2. The three lines P_1, P_2, and P_3 are mutually perpendicular, and the controller P consists simply of four particles $pj, j = 1,2,3,4$, each of mass m, which are constrained to move along P_1 and P_2 in such a manner that they always lie at the vertices of a rhombus with sides of constant length l. Thus, P_1, P_2, and P_3 are principal axes of P for its mass center P^*, and the corresponding moments of inertia are $2ml_2{}^2$, $2ml_1{}^2$, and $2ml^2$, respectively, where $l_1{}^2 + l_2{}^2 = l^2$. When P assumes a square configuration ($l_1 = l_2$), its inertia ellipsoid is symmetric about P_3, and consequently it experiences no gravity torque about P_3 regardless of its orientation. This, then, is the null or inactive position of the controller. For any nonsquare (active) configuration the inertia ellipsoid is nonsymmetric, and the gravity torque will, in general, have a component along P_3. Thus, by controlling the configuration of P it is possible to control the magnitude and sense of the P_3 component of the gravity torque. Note also that the moment of inertia about P_3 remains constant ($2ml^2$) regardless of the configuration of P, so that, while variations in the configuration alter the gravity torque, they do not affect the inertia torque about P_3.

The complete system is pictured in Fig. 3. It consists of the satellite main body B, two identical roll-yaw dampers $D1$ and $D2$, and the pitch controller P. The dampers are symmetrical rigid bodies (rotors) with axial moments of inertia J, and they are connected to the main body in such a way that 1) the mass centers $D1^*$ and $D2^*$ of $D1$ and $D2$, respectively, are fixed in B, 2) the symmetry axes of $D1$ and $D2$ are mutually perpendicular and fixed in B, 3) $D1$ and $D2$ are free to rotate relative to the main body about their respective symmetry axes, 4) rotations of $D1$ and $D2$ relative to the main body are resisted by torques proportional to respective relative angular velocities, d being the constant of proportionality, and 5) the system BD consisting of B, $D1$ and $D2$ has an axisymmetric inertia ellipsoid for its mass center BD^*, the axis of symmetry being perpendicular to the symmetry axes of $D1$ and $D2$. The axial and transverse moments of inertia of BD for BD^* are I_3 and I_1, respectively, and they are not affected by the relative motions of the symmetric dampers.

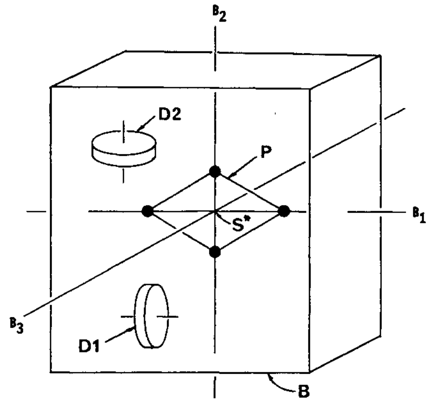

Fig.3 Schematic of the satellite system.

In order to describe the location of the pitch controller, it is convenient to first introduce three mutually perpendicular lines B_1, B_2, and B_3 fixed in B, each passing through BD^* with B_1 and B_2 parallel to the symmetry axes of $D1$ and $D2$, respectively. From this point on, the term pitch axis refers to B_3 and roll-yaw axes to B_1 and B_2, unless otherwise stated. The pitch controller can now be positioned in B by aligning P_1, P_2, and P_3 (see Fig. 2) with B_1, B_2, and B_3, respectively, such that P^* and BD^* coincide. As a consequence, the mass center S^* of the complete satellite system S, consisting of B, $D1$, $D2$, and P, also coincides with P^*, and BD^* and is thus fixed in B. It follows that, when P is in its null configuration ($l_1 = l_2$), the inertia ellipsoid of S for S^* is symmetric about B_3, and when P is in an active configuration ($l_1 \neq l_2$), B_1 and B_2 are principal axes of S for S^*.

Equations of Motions

With a system selected, the next task is to construct a set of linear differential equations from which the stability of the desired satellite attitude motion can be ascertained. In order to proceed, it is necessary to select quantities which can be used to describe the attitude motions of the main satellite body and the dampers.

Consider the main body B first. For problems involving gravity torque it is advantageous to employ variables which describe the orientation of the satellite in an orbital frame A. Three mutually perpendicular axes A_1, A_2, and A_3 (see Fig. 1) are fixed in A and intersect at S^*; A_1 is parallel to the line joining S^* and E^*, A_3 is normal to the orbit plane, and A_2 is the common perpendicular to A_1 and A_3. The angle θ describes the orientation of A in the Newtonian frame. The orientation of B in A is described by the angles θ_1, θ_2, and θ_3 pictured in Fig. 4 and described as follows: align the body-fixed axes B_1, B_2, and B_3 with the orbital axes A_1, A_2, and A_3, respectively, and then perform successive right-handed rotations of B of

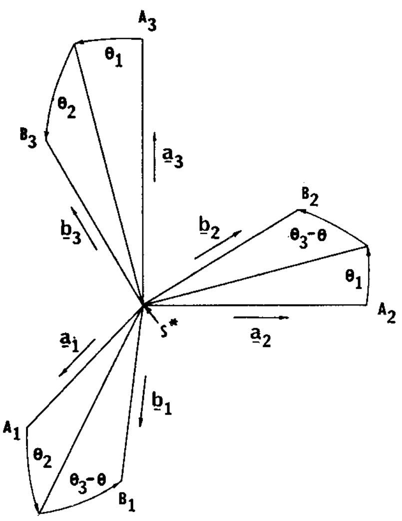

Fig. 4 Attitude angles.

amounts θ_1 about $\mathbf{b}_1$, θ_2 about $\mathbf{b}_2$, and $(\theta_3 - \theta)$ about $\mathbf{b}_3$. If the axes B_i and A_i, $i = 1,2,3$, are aligned when $\theta = 0$, the desired motion of B then corresponds to $\theta_1 = \theta_2 = \theta_3 = 0$.

Consider next the dampers $D1$ and $D2$. Their motion is most easily referred to the body B, and the angular velocities of $D1$ and $D2$ in B may be expressed as $\omega_1\mathbf{b}_1$ and $\omega_2\mathbf{b}_2$, respectively. Because the dampers rotate about their symmetry axes, their actual angular displacements with respect to B do not appear in the equations of motion.

Because of the constraints on the motions of the particles, only one coordinate, say l_1 (see Fig. 2), is needed to completely describe the configuration of the pitch controller P. However, l_1 will be eliminated from the equations by introducing a feedback control law for the pitch controller.

The differential equations governing the behavior of the satellite can now be derived by applying the angular momentum principle first to the entire system S and then to the individual dampers. In the analysis which follows, only a linearized version of the equations is utilized. Consequently, since the full nonlinear equations are quite long, they are not included here.

In the proposed control scheme, the function of the controller P is to control only the pitch motion of the satellite because it is assumed that the roll-yaw motion will remain small due to the natural restoring action of the gravity torque. As a consequence, selection of the control law for P is based on the assumptions that there is no roll-yaw motion of the main body B and that the roll-yaw dampers $D1$ and $D2$ are fixed in B, i.e., that θ_1, θ_2, ω_1, and ω_2 are all equal to zero. For these conditions, the equations of motion are all satisfied identically with exception of the pitch equation which becomes

$$(I_3 + 2ml^2)\ddot{\theta}_3 + 3n^2m(l_1^2 - l_2^2)\sin[2(\theta_3 - \theta)] = 0 \quad (3)$$

The second term in this equation is the component of the gravity torque acting on the controller about the pitch axis, and the only factor in this term which can be varied by distorting P is $(l_1^2 - l_2^2)$. If the gravity torque is to act as an effective pitch control torque, the control law governing $(l_1^2 - l_2^2)$ must be such that the gravity torque tends to drive θ_3 and $\dot{\theta}_3$ toward zero regardless of the value of θ, and the control law must not violate the constraint that $l_1^2 + l_2^2 = l^2$. In addition, to simplify the analysis, it is desirable for the resulting control torque to be continuous and linear in θ_3 and $\dot{\theta}_3$ for small values of θ_3 and $\dot{\theta}_3$. All of these conditions are satisfied by the control law

$$l_1^2 - l_2^2 = l^2 \sin[2(\theta_3 - \theta)]\sin(k_1\theta_3 + k_2\dot{\theta}_3) \quad (4)$$

where k_1 and k_2 are the attitude and rate feedback gains, respectively. When this control law is inserted into Eq. (3) and the resulting equation is linearized in θ_3, the pitch equation becomes

$$(I_3 + 2ml^2)\ddot{\theta}_3 + 3n^2(ml^2)\sin^2(2\theta)(k_1\theta_3 + k_2\dot{\theta}_3) = 0 \quad (5)$$

The coefficients in this equation are periodic, but, for positive values of k_1 and k_2, they are non-negative.

When the system S has the desired motion, the orientation of B is fixed in N. In terms of the variables of interest, this motion corresponds to $\theta_1 = \theta_2 = \theta_3 = 0$. When these conditions are substituted into the full nonlinear attitude equations (after incorporation of the control law), the resulting equations are satisfied if, and only if, $\omega_1 = \omega_2 = 0$. That is, B can remain fixed in N only when the dampers are at rest. Thus, the null or zero solution of the attitude equations is the only solution which corresponds to the desired motion.

In the next section, the stability of the zero solution of the attitude equations is determined by analyzing a set of variational equations derived from the attitude equations by linearizing about the equilibrium solution, that is, by neglecting all terms of second and higher degrees in θ_1, θ_2, θ_3, ω_1, and ω_2 and their derivatives. Before presenting these equations it is convenient to simplify notation by defining the five parameters

$$\begin{aligned} &Q = J/(I_1 + ml^2), \quad \Delta = d/nJ, \quad R = (I_3 + 2ml^2)/(I_1 + ml^2) \\ &K_1 = 3ml^2k_1/2(I_3 + 2ml^2), \quad K_2 = 3ml^2nk_2/2(I_3 + 2ml^2) \end{aligned} \quad (6)$$

It is also useful to introduce new variables w_1, w_2, and τ defined as follows:

$$w_1 = \omega_1 \cos\theta + \omega_2 \sin\theta, \quad w_2 = -\omega_1 \sin\theta + \omega_2 \cos\theta \quad (7)$$

$$\tau = nt \quad (8)$$

Then, letting primes denote differentiation with respect to τ, the variational equations may be written as follows:

$$\begin{aligned} \theta_1'' &= \theta_1 + 2\theta_2' + [\Delta Q/(1 - Q)]w_1 \\ \theta_2'' &= [1 - 3(R - 1)/(1 - Q)]\theta_2 - 2\theta_1' + [\Delta Q/(1 - Q)]w_2 \\ w_1' &= -[\Delta/(1 - Q)]w_1 + w_2 \\ w_2' &= 3[(R - 1)/(1 - Q)]\theta_2 - w_1 - [\Delta/(1 - Q)]w_2 \end{aligned} \quad (9)$$

$$\theta_3'' = -[1 - \cos(4\tau)](K_2\theta_3' + K_1\theta_3) \quad (10)$$

Since Eqs. (9) contain only the roll-yaw variables θ_1, θ_2, w_1, and w_2 and Eq. (10) contains only the pitch variable θ_3, the pitch and roll-yaw variational equations are uncoupled. This indicates that in the linear analysis the motion of the pitch controller does not affect the roll-yaw motion. In other words, the roll-yaw equations are the same as if the combination of the main body B and the controller P were a single symmetric rigid body with axial and transverse principal centroidal moments of inertia of $(I_3 + 2ml^2)$ and $(I_1 + ml^2)$, respectively.

In the pitch equation the coefficients are periodic functions of τ, of period $\pi/2$, whereas the roll-yaw equations have constant coefficients. It was for the purpose of obtaining constant coefficients in the roll-yaw equations that the variables w_1 and w_2 were introduced. Because of the linear relationship between w_i and ω_i, $i = 1,2$, the null solution of Eqs. (9) and (10) corresponds to the desired satellite motion. The stability of this solution is discussed next.

Stability Analyses

To establish the theoretical feasibility of the proposed control scheme, it is necessary to demonstrate that, for some set of system parameters, the desired attitude motion of the system is a stable motion. In order to do so, one must first define stability as it applies to this analysis. As already pointed out, the desired motion corresponds to the conditions $\theta_1 = \theta_2 = \theta_3 = 0$. This motion is called stable if, given an arbitrarily small number, one can restrict $|\theta_i|$, $|\dot{\theta}_i|$, $i = 1,2,3$, and $|\omega_i|$, $i = 1,2$, at $t = t_0$ to values so small that $|\theta_i(t)|$ remains smaller than this number for all $t > t_0$; the motion is called asymptotically stable if, under the same conditions, $\theta_i(t) \to 0$ as $t \to \infty$; otherwise, it is said to be unstable.

Because the desired satellite motion is described by the null solution of the attitude equations, "stability of the motion" is synonymous with stability of this solution; and although the full equations are nonlinear, the stability of their null solution can be determined for a wide range of cases by examining the associated linearized variational equations. This, in turn, is facilitated by the fact that the linearized pitch and roll-yaw equations are uncoupled, which makes it possible to analyse the stability of the solutions of the two corresponding subsets of equations independently.

The techniques which will be used for determining the stability of the null solution of the linearized variational equations require that the equations be written in the form of a set of first-order differential equations which, in matrix notation, may be expressed as $x' = Mx$ where x is a column matrix whose elements are the variables of interest, M is a square matrix of coefficients, and the null solution of this equation corresponds to the desired motion. In the analyses which follow, the elements of M will be of two types, constants and periodic functions of τ, of period $\pi/2$.

When all the elements of the matrix M are constants, the stability of the motion is determined by examining the characteristic values of M.[5] The motion is asymptotically stable if, and only if, the real parts of all the characteristic values are negative, and it is unstable if the real part of any characteristic value is positive.

When the elements of M are periodic functions of τ, the stability of the motion can be established by making use of Floquet Theory.[6] In this case, stability is determined by examining the characteristic values of a square matrix $Y(\pi/2)$, where $Y(\tau)$ satisfies the matrix differential equation

$$Y'(\tau) = M(\tau)Y(\tau) \tag{11}$$

and the initial conditions

$$Y(0) = I \tag{12}$$

where I is the identity matrix. The motion is asymptotically stable if, and only if, the moduli of all the characteristic values of $Y(\pi/2)$ are less than unity, and it is unstable if any characteristic value has a modulus greater than unity. In order to evaluate the elements of the matrix $Y(\pi/2)$, the variational equations are integrated numerically over the interval $0 \leq \tau \leq \pi/2$ with the initial conditions given in Eq. (12).

The linearized roll-yaw equations are Eqs. (9). This system of equations is of sixth order, and the coefficients depend upon the three satellite parameters Q, Δ, and R. In discussing the effects of the parameters on stability, it is useful to think of Q as a measure of the relative size of the dampers, of Δ as a measure of viscosity, and of R as the ratio of axial to transverse moments of inertia.

Equations (9) can be written in the desired form $x' = Mx$ by defining the elements of the column matrix x as

$$x_1 = \theta_1, \quad x_2 = \theta_2, \quad x_3 = \theta_1', \quad x_4 = \theta_2', \quad x_5 = w_1, \quad x_6 = w_2$$

and if new parameters D_1, D_2, and D_3 are defined as

$$D_1 = \Delta/(1 - Q), \quad D_2 = \Delta Q/(1 - Q), \quad D_3 = 3(R - 1)/(1 - Q) \tag{13}$$

then, the characteristic equation associated with the matrix M takes the form

$$\lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 = 0 \tag{14}$$

where

$$\begin{gathered} a_1 = 2D_1, \quad a_2 = (3 + D_1^2) + D_3, \quad a_3 = 4D_1 + (2D_1 - D_2)D_3 \\ a_4 = (3 + 2D_1^2) + D_1(D_1 - D_2)D_3 \\ a_5 = 2D_1 + (3D_2 - 2D_1)D_3 \\ a_6 = (1 + D_1^2) + (D_1D_2 - D_1^2 - 1)D_3 \end{gathered} \tag{15}$$

(The parameters Q, Δ, and R are a natural selection for describing the satellite, whereas D_1, D_2, and D_3 are more convenient for dealing with the equations of motion.)

In order to determine the stability of the roll-yaw motion, it is only necessary to know the sign of the real part of each root of Eq. (14), which can be determined by applying the Routh-Hurwitz stability criterion for a sixth-order system[7]: for all the roots to have negative real parts, it is necessary and sufficient that 1) all the coefficients be positive, i.e.,

$$a_i > 0, \; i = 1,2,3,4,5,6 \tag{16}$$

and 2) the following two auxiliary conditions be satisfied simultaneously:

$$\begin{gathered} f_1 = a_3(a_1a_2 - a_3) - a_1(a_1a_4 - a_5) > 0 \\ f_2 = (a_1a_2 - a_3)[a_5(a_4a_3 - a_2a_5) + a_6(2a_1a_5 - a_3^2)] + \\ (a_1a_4 - a_5)[a_1a_3a_6 - a_5(a_1a_4 - a_5) - a_1^3a_6^2] > 0 \end{gathered} \tag{17}$$

When the inequality signs in Eqs. (16) and (17) are replaced with equality signs, the resulting equations define surfaces or boundaries in the R-Δ-Q parameter space. For the purpose of describing these boundaries, it is convenient to solve the equations for R as a function of Q and Δ. To determine the location of the stable regions, it is necessary to consider the ranges of permissible parameter values, restricted as follows:

$$R > 0, \quad \Delta > 0, \quad 0 < Q < \tfrac{2}{3} \tag{18}$$

The upper limit of Q is $\frac{2}{3}$ because the two roll-yaw dampers, $D1$ and $D2$, are assumed identical. This value is approached when the masses of B and P are reduced to zero and $D1$ and $D2$ take on the shape of thin discs.

Except for a_1, which is independent of R and always positive, all the coefficients are linear in D_3; and consequently, linear in R. The regions in the parameter space where the coefficients are positive are obtained by substituting from Eqs. (13) into Eq. (15) to get expressions for a_i in terms of R, Δ, and Q and then substituting these expressions into Eq. (16). The regions thus obtained are summarized in Table 1. (The numerical limits indicated in the table are used to determine the stability boundaries which result when all conditions are considered.)

In treating the auxiliary conditions of Eq. (17), it is convenient to define the associated regions first in terms of D_1, D_2, and D_3 and then to convert to R, Δ, and Q. To do so, it is necessary to utilize the inequalities

$$D_1 > 0, \quad D_2 > 0, \quad D_1 > D_2 \tag{19}$$

which follow from Eq. (18) and (13).

Table 1 Regions in the roll-yaw parameter space defined by the Routh-Hurwitz stability criteria for the coefficients in the characteristic polynomial

Condition	Region
$a_2 > 0$	$R > Q - \Delta^2/3(1 - Q) < 1$
$a_3 > 0$	$R > (\frac{1}{3})(2 + Q)/(2 - Q) < 1$
$a_4 > 0$	$R > \frac{1}{3} - (1 - Q)^2/\Delta^2 < \frac{1}{3}$
$a_5 > 0$	$R < (\frac{4}{3})[1 - (11/8)Q]/[1 - (3/2)Q] > \frac{4}{3}$
$a_6 > 0$	$R < (\frac{4}{3}) \dfrac{\{(1 + \Delta^2) - Q[1 + (1 - Q)/4]\}}{[(1 + \Delta^2) - Q]} \begin{matrix} < \frac{4}{3} \\ > 1 \end{matrix}$

Substituting from Eq. (15) into the expression for the function f_1 in the first auxiliary condition of Eq. (17) gives

$$f_1 = D_3[D_2(2D_1 - D_2)D_3 + 2D_1D_2(4 + D_1^2)] \quad (20)$$

Treated as a polynomial in D_3, f_1 is of second degree, and the roots of f_1 are

$$D_3 = 0, \quad D_3 = -2D_1(4 + D_1^2)/(2D_1 - D_2) < 0 \quad (21)$$

In addition

$$df_1/dD_3|_{D_3=0} = 2D_1D_2(4 + D_1^2) > 0 \quad (22)$$

and, consequently, f_1 is negative for values of D_3 between those given in Eq. (21), and positive elsewhere.

The function f_2 in the second auxiliary condition of Eq. (17), when expressed in terms of D_1, D_2, and D_3, is

$$f_2 = D_3^3\{2D_2^2[2D_1(2 + D_1^2) - (4 + 3D_1^2)D_2 + D_1D_2^2]D_3 + 4D_2^2(D_1^2 + 4)^2(D_1 - D_2)\} \quad (23)$$

which is a fourth-degree polynomial in D_3, with roots

$$D_3 = 0 \text{ (triple root)}$$

$$D_3 = -\frac{2(D_1^2 + 4)^2(D_1 - D_2)}{2D_1(2 + D_1^2) - (4 + 3D_1^2)D_2 + D_1D_2^2} \quad (24)$$

The numerator of the nonzero root is positive because $D_1 > D_2$. To show that the denominator is also positive, it is convenient to replace D_2 with QD_1 [see Eq. (13)], after which the denominator can be expressed as $D_1(1-Q)[4+D_1^2(2-Q)]$, which is positive. Consequently, the nonzero root is negative. In addition, the first of the derivatives of f_2 with respect to D_3 which is nonzero for $D_3 = 0$ is the third derivative, which is given by

$$d^3f_2/dD_3^3|_{D_3=0} = 24D_2^2(D_1^2 + 4)^2(D_1 - D_2) > 0 \quad (25)$$

Now, because f_2 has only two distinct roots, the sign of f_2 changes only twice, and in view of Eq. (25), it must be negative for values of D_3 between the two roots and positive elsewhere.

The regions bounded by the values given in Eqs. (21) and (24) are expressed in terms of R, Δ, and Q by substituting from Eq. (13), and the results are shown in Table 2.

When the requirements that a_3 and f_1 be positive are combined, the entire region $0 < R < 1$ is excluded because

$$(2 + Q) > (-2 + 5Q) > (-2 + 5Q) - 2\Delta^2/(1 - Q) \quad (26)$$

Since the boundaries resulting from the condition on a_2, a_4, and f_2 all lie within this region, they do not create additional stability boundaries. In addition, the boundary for $a_5 > 0$ falls within the region excluded by the condition $a_6 > 0$. Consequently, the stable region is completely defined by the conditions on a_3, a_6, and f_1 and is given by

$$1 < R < (\tfrac{4}{3})\{(1 + \Delta^2) - Q[1 + (1 - Q)/4]\}/[(1 + \Delta^2) - Q] \quad (27)$$

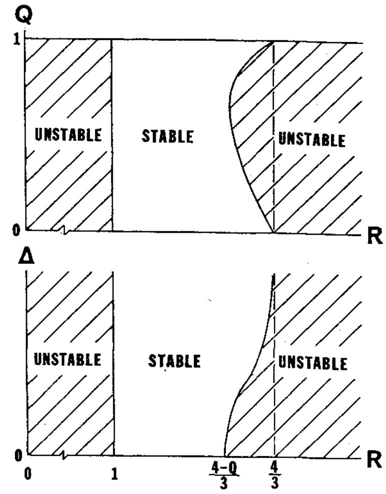

Fig. 5 Roll-yaw stability charts.

The stability boundaries in the $R - \Delta$ space are of the form shown in Fig. 5. As Δ increases, the range of stable values of R increases, and when Δ approaches infinity, the range becomes $1 < R < \frac{4}{3}$. This upper limit coincides with the region of infinitesimal stability for the rigid body without dampers.[4] With the dampers added, however, this region is asymptotically stable.

The form of the stability boundaries in the $R - Q$ space is also shown in Fig. 5. In this case, the effect of a change in the value of Q on the range of stable values of R depends upon the magnitude of Q. From a practical standpoint, however, it is desirable to keep the dampers as small as possible, and for small values of Q an increase in damper size results in a decrease in the range of stable values of R.

The linearized pitch equation is Eq. (10). To write it in the form $x' = Mx$, the elements of x are defined as $x_1 = \theta_3$, $x_2 = \theta_3'$ and the corresponding matrix M is given by

$$M = \begin{bmatrix} 0 & 1 \\ -K_1[1 - \cos(4\tau)] & -K_2[1 - \cos(4\tau)] \end{bmatrix} \quad (28)$$

Since the coefficients are periodic, the stability analysis requires the application of Floquet theory.

A Floquet analysis was carried out for a range of values of the two parameters K_1 and K_2, and the results are shown in Fig. 6. For a second-order system with constant coefficients, the entire first quadrant of the $K_1 - K_2$ space corresponds to asymptotically stable motion. The effect of the periodic components of the coefficients is the creation of small regions of instability in the first quadrant, near the K_1 axis.

The analyses of this section show that, for the proposed satellite system, there exist regions in the system parameter space for which both the desired pitch and roll-yaw motions are asymptotically stable. The conclusion to be drawn from these results is that it is theoretically possible to utilize the gravity torque as a control torque for the complete three-axis attitude stabilization of a satellite with a fixed inertial orientation.

Table 2 Regions in the roll-yaw parameter space defined by the auxiliary Routh-Hurwitz stability criteria

Condition	Region
$f_1 > 0$	$R > 1$
	$R < (\frac{1}{3})(-2 + 5Q)/(2 - Q) - 2\Delta^2/3(2 - Q)(1 - Q) < 1$
$f_2 > 0$	$R > 1$
	$R < 1 - (\frac{2}{3})\{[\Delta^2 + 4(1 - Q)^2]^2\}/[\Delta^3(1 - Q)(2 - Q) + 4(1 - Q)^3] < 1$

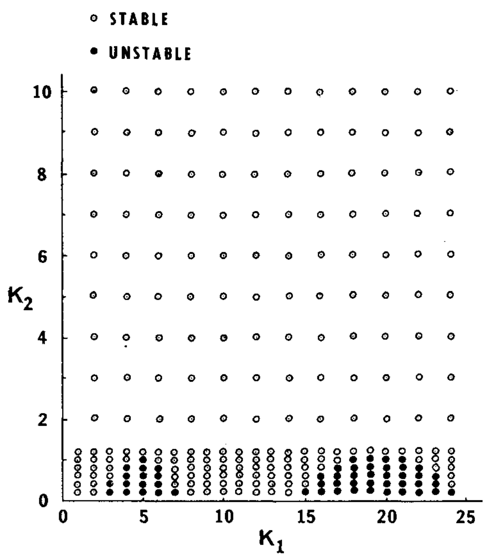

Fig. 6 Pitch stability chart.

Discussion

Having established the theoretical feasibility of the proposed concept we turn to consideration of the suitability of the technique for specific applications. This will depend upon a variety of factors, some of which, such as performance capability, control system implementation, and effects of orbital eccentricity, have already been investigated. A brief summary of the results of these studies follows.

The stability analysis of the previous section identifies regions in the system parameter space for which the desired attitude motion is asymptotically stable. Although stability is guaranteed whenever the values of the parameters are selected from within these regions, the performance of the control system depends upon the particular set of values selected. One important measure of system performance is the length of time required for attitude errors resulting from disturbances to disappear. To quantify this performance measure, it is useful to introduce a quantity τ_s, called the settling time, which is defined as the time necessary for transients resulting from a small disturbance to be reduced by a factor of at least $1/e$, where e is the base of the natural logarithm.

For roll-yaw motions, τ_s is determined by the location in the complex plane of that root of the associated characteristic equation [Eq. (14)] which lies nearest to the imaginary axis. The effects of the three system parameters R, Q, and Δ on the location of this root were determined numerically. The results show that the settling time is minimized by selecting a value of the inertia ratio R as close as is practicable to the upper limit of its stable range and by making the damper size Q as large as possible. For given values of R and Q, there is a well defined optimum value of damper viscosity Δ. The lower limit on τ_s is dictated largely by the upper limit on damper size, which will be determined by considerations beyond the scope of this analysis, such as size and weight restrictions. For values of Q as large as 0.1, the minimum value of τ_s is approximately ten orbital periods. The adequacy of settling times of this duration will depend upon the particular application.

For pitch motions, τ_s is determined by the characteristic value of the association Y matrix [see Eq. (11)] which lies nearest the unit circle in the complex plane. The dependence of τ_s on the parameters K_1 and K_2 is complicated. Generally speaking, τ_s decreases as K_1 and K_2 increase. For pitch motions, values of τ_s of substantially less than one orbital period can be obtained, although there may be no advantage in having the pitch performance exceed the roll-yaw performance.

As regards control system implementation, one may contemplate a number of schemes different from the one already discussed. Two additional alternatives have been found to be effective. In one, pitch damping was obtained by using a viscous damper rather than rate feedback. In the other, three axis damping was provided by a single spherical viscous damper. When viscous damping is used for pitch, the performance capability in terms of settling times is similar to that for roll-yaw motions.

Finally, effects of orbital eccentricity were studied both to determine whether small deviations from a nominally circular orbit affect stability and to see whether or not the concept can be utilized for highly eccentric orbits. For elliptic orbits, both the pitch and the roll-yaw variational equations turn out to have periodic coefficients, so that now Floquet analysis is required in both cases. The results of such analysis show that orbital eccentricity has very little effect on pitch stability, whereas, for roll-yaw motions, the stable region of the satellite parameter space shrinks as eccentricity increases. Stable regions still exist, however, for eccentricities as high as 0.5. Hence, it may be concluded that the concept is applicable for most of the orbits one may expect to encounter in practice.

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